Proof Details for “Performance Analysis of Godard-Based Blind Channel Identification”†

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I. Proof of Lemma 1

The expected block-ASPE can be written

$$\begin{aligned}
\mathrm{E}\{\mathcal{E}_{\hat{\underline{\boldsymbol{h}}}}\} &= \mathrm{E}\left\{\frac{1}{Q+1}\sum_{\delta=0}^{Q}\left\|\theta_{\hat{\underline{\boldsymbol{h}}}}\hat{\hat{\mathbf{h}}}^{(0)}_{\nu+M-\delta} - \mathbf{h}^{(0)}_{\nu+M-\delta}\right\|_2^2\right\} \\
&= \mathrm{E}\left\{\frac{1}{Q+1}\sum_{\delta=0}^{Q}\left(\left\|\theta_{\hat{\underline{\boldsymbol{h}}}}\hat{\hat{\mathbf{h}}}^{(0)}_{\nu+M-\delta} - \theta_{\hat{\underline{\boldsymbol{h}}}}\,\mathrm{E}\Big\{\hat{\hat{\mathbf{h}}}^{(0)}_{\nu+M-\delta}\Big\}\right\|_2^2 + \left\|\theta_{\hat{\underline{\boldsymbol{h}}}}\,\mathrm{E}\Big\{\hat{\hat{\mathbf{h}}}^{(0)}_{\nu+M-\delta}\Big\} - \mathbf{h}^{(0)}_{\nu+M-\delta}\right\|_2^2\right)\right\}
\end{aligned}$$

where $\theta_{\hat{\underline{\boldsymbol{h}}}}$ was specified in the text as

$$\theta_{\hat{\underline{\boldsymbol{h}}}} = \arg\min_{\theta\in\mathbb{C}}\left\|\theta\hat{\underline{\boldsymbol{h}}}^{(0)}_{\nu+M} - \underline{\boldsymbol{h}}^{(0)}_{\nu+M}\right\|_2^2. \tag{1}$$

Expanding (1) and zeroing the partial derivative with respect to θ, it is straightforward to show that

$$\theta_{\hat{\underline{\boldsymbol{h}}}} = \frac{\hat{\underline{\boldsymbol{h}}}^{(0)H}_{\nu+M}\underline{\boldsymbol{h}}^{(0)}_{\nu+M}}{\|\hat{\underline{\boldsymbol{h}}}^{(0)}_{\nu+M}\|_2^2}. \tag{2}$$

Recalling

$$\hat{\hat{\mathbf{h}}}^{(0)}_{\nu+M-\delta} := \frac{1}{N}\sum_{n=0}^{N-1}\mathbf{r}_{n-\delta}y^*_{n-M}$$

it can be seen that

$$\mathrm{E}\Big\{\hat{\hat{\mathbf{h}}}^{(0)}_{\nu+M-\delta}\Big\} = \mathrm{E}\{\mathbf{r}_{n-\delta}y^*_{n-M}\} = \hat{\mathbf{h}}^{(0)}_{\nu+M-\delta},$$

and thus

$$\begin{aligned}
\mathrm{E}\{\mathcal{E}_{\hat{\underline{\boldsymbol{h}}}}\} &= \frac{1}{Q+1}\sum_{\delta=0}^{Q}\left(|\theta_{\hat{\underline{\boldsymbol{h}}}}|^2\,\mathrm{E}\left\{\left\|\hat{\hat{\mathbf{h}}}^{(0)}_{\nu+M-\delta} - \hat{\mathbf{h}}^{(0)}_{\nu+M-\delta}\right\|_2^2\right\} + \left\|\theta_{\hat{\underline{\boldsymbol{h}}}}\hat{\mathbf{h}}^{(0)}_{\nu+M-\delta} - \mathbf{h}^{(0)}_{\nu+M-\delta}\right\|_2^2\right) \\
&= \mathcal{E}_{\hat{\underline{\boldsymbol{h}}}} + \frac{|\theta_{\hat{\underline{\boldsymbol{h}}}}|^2}{Q+1}\sum_{\delta=0}^{Q}\mathrm{E}\left\{\left\|\hat{\hat{\mathbf{h}}}^{(0)}_{\nu+M-\delta} - \hat{\mathbf{h}}^{(0)}_{\nu+M-\delta}\right\|_2^2\right\}
\end{aligned} \tag{3}$$

The bulk of the proof is spent analyzing the righthand term above.

$$\begin{aligned}
\mathrm{E}\Big\{\|\hat{\hat{\mathbf{h}}}^{(0)}_i - \hat{\mathbf{h}}^{(0)}_i\|_2^2\Big\} &= \mathrm{E}\Big\{\|\hat{\hat{\mathbf{h}}}^{(0)}_i\|_2^2 - (\hat{\hat{\mathbf{h}}}^{(0)}_i)^H\hat{\mathbf{h}}^{(0)}_i - (\hat{\mathbf{h}}^{(0)}_i)^H\hat{\hat{\mathbf{h}}}^{(0)}_i + \|\hat{\mathbf{h}}^{(0)}_i\|_2^2\Big\} \\
&= \mathrm{E}\Big\{\|\hat{\hat{\mathbf{h}}}^{(0)}_i\|_2^2\Big\} - \mathrm{E}\Big\{\hat{\hat{\mathbf{h}}}^{(0)}_i\Big\}^H\hat{\mathbf{h}}^{(0)}_i - (\hat{\mathbf{h}}^{(0)}_i)^H\,\mathrm{E}\Big\{\hat{\hat{\mathbf{h}}}^{(0)}_i\Big\} + \|\hat{\mathbf{h}}^{(0)}_i\|_2^2 \\
&= \mathrm{E}\Big\{\|\hat{\hat{\mathbf{h}}}^{(0)}_i\|_2^2\Big\} - \|\hat{\mathbf{h}}^{(0)}_i\|_2^2.
\end{aligned} \tag{4}$$

Now we wish to examine the lefthand term in (4) for $i = \nu + M - \delta$. To avoid ugly notation, we examine the scalar-valued channel case, i.e., $P = 1$. The extension to general P is trivial. Thus we have

$$
\begin{aligned}
\mathrm{E}\Big\{|\hat{\hat{h}}^{(0)}_{\nu+M-\delta}|^2\Big\} &= \mathrm{E}\left\{\left|\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k,j} h_j^{(k)} s^{(k)}_{n-\delta-j}\sum_{\ell,i} q_i^{(\ell)*} s^{(\ell)*}_{n-M-i}\right|^2\right\} \\
&= \frac{1}{N^2}\sum_{k,j,\ell,i} h_j^{(k)} q_i^{(\ell)*}\sum_{a,b,c,d} h_b^{(a)*} q_d^{(c)}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\underbrace{\mathrm{E}\big\{s^{(k)}_{n-\delta-j}s^{(\ell)*}_{n-M-i}s^{(a)*}_{m-\delta-b}s^{(c)}_{m-M-d}\big\}}_{B}
\end{aligned}
$$

Let's consider the quantity B for various (mutually-exclusive) cases:

A.1. $\underline{k=\ell=a=c}$

A.1.1. $\left\{\begin{array}{ll} & n-\delta-j \;=\; n-M-i \;=\; m-\delta-b \;=\; m-M-d \\ \Leftrightarrow & j=M-\delta+i \;\text{ and }\; b=M-\delta+d \;\text{ and }\; m=d+n-i \end{array}\right\}, B=\mathrm{E}\{|s_i^{(k)}|^4\}=\kappa_s^{(k)}\sigma_s^4$

A.1.2. $\left\{\begin{array}{ll} & n-\delta-j \;=\; n-M-i \;\neq\; m-\delta-b \;=\; m-M-d \\ \Leftrightarrow & j=M-\delta+i \;\text{ and }\; b=M-\delta+d \;\text{ and }\; m\neq d+n-i \end{array}\right\}, B=\mathrm{E}^2\{|s_i^{(k)}|^2\}=\sigma_s^4$

A.1.3. $\left\{\begin{array}{ll} & n-\delta-j \;=\; m-\delta-b \;\neq\; n-M-i \;=\; m-M-d \\ \Leftrightarrow & j\neq M-\delta+i \;\text{ and }\; d=m-n+i \;\text{ and }\; b=m-n+j \end{array}\right\}, B=\mathrm{E}^2\{|s_i^{(k)}|^2\}=\sigma_s^4$

A.1.4. $\left\{\begin{array}{ll} & n-\delta-j \;=\; m-M-d \;\neq\; n-M-i \;=\; m-\delta-b \\ \Leftrightarrow & j\neq M-\delta+i \;\text{ and }\; d=m-n-M+\delta+j \;\text{ and }\; b=m-n+M-\delta+i \end{array}\right\}, B=\mathrm{E}^2\{(s_i^{(k)})^2\}=\sigma_s^4 \;\text{ iff }\; s\in\mathbb{R}$

A.1.5. Else $B=0$

A.2. $\underline{k=\ell\neq a=c}$

A.2.1. $\left\{\begin{array}{ll} & n-\delta-j \;=\; n-M-i \;\text{ and }\; m-\delta-b \;=\; m-M-d \\ \Leftrightarrow & j=M-\delta+i \;\text{ and }\; b=M-\delta+d \end{array}\right\}, B=\mathrm{E}\{|s_i^{(k)}|^2\}\,\mathrm{E}\{|s_i^{(a)}|^2\}=\sigma_s^4$

A.2.2. Else $B=0$

A.3. $\underline{k=a\neq\ell=c}$

A.3.1. $\left\{\begin{array}{ll} & n-\delta-j \;=\; m-\delta-b \;\text{ and }\; n-M-i \;=\; m-M-d \\ \Leftrightarrow & d=m-n+i \;\text{ and }\; b=m-n+j \end{array}\right\}, B=\mathrm{E}\{|s_i^{(k)}|^2\}\,\mathrm{E}\{|s_i^{(\ell)}|^2\}=\sigma_s^4$

A.3.2. Else $B=0$

A.4. $\underline{k=c\neq\ell=a}$

A.4.1. $\left\{\begin{array}{ll} & n-\delta-j \;=\; m-M-d \;\text{ and }\; n-M-i \;=\; m-\delta-b \\ \Leftrightarrow & d=m-n-M+\delta+j \;\text{ and }\; b=m-n+M-\delta+i \end{array}\right\}, B=\mathrm{E}\{(s_i^{(k)})^2\}\,\mathrm{E}\{(s_i^{(\ell)})^2\}=\sigma_s^4 \;\text{ iff }\; s\in\mathbb{R}$

A.4.2. Else $B=0$

A.5. <u>all other cases</u>: $B=0$.

Plugging in non-zero B from the cases A.1.-A.5. above, we have

$$
\begin{aligned}
& N^2 \, \mathrm{E}\Big\{|\hat{\hat{h}}^{(0)}_{\nu+M-\delta}|^2\Big\} \\
= & \underbrace{\sum_k \left(\sum_i h^{(k)}_{M-\delta+i} q^{(k)*}_i\right) \left(\sum_d h^{(k)*}_{M-\delta+d} q^{(k)}_d\right) \sum_{n=0}^{N-1} \kappa^{(k)}_s \sigma^4_s}_{A.1.1.} \\
& + \underbrace{\sum_k \left(\sum_i h^{(k)}_{M-\delta+i} q^{(k)*}_i\right) \left(\sum_d h^{(k)*}_{M-\delta+d} q^{(k)}_d\right) \sum_{n=0}^{N-1} \sum_{m\neq d+n-i}^{N-1} \sigma^4_s}_{A.1.2.} \\
& + \underbrace{\sum_{n,m=0}^{N-1} \sum_k \left(\sum_i q^{(k)*}_i q^{(k)}_{m-n+i}\right) \left(\sum_{j\neq M-\delta+i} h^{(k)}_j h^{(k)*}_{m-n+j}\right) \sigma^4_s}_{A.1.3.} \\
& + \underbrace{\sum_{n,m=0}^{N-1} \sum_k \left(\sum_i h^{(k)*}_{m-n+M-\delta+i} q^{(k)*}_i\right) \left(\sum_{d\neq m-n+i} h^{(k)}_{n-m+M-\delta+d} q^{(k)}_d\right) \sigma^4_s}_{A.1.4.\text{—only exists iff } s\in\mathbb{R}} \\
& + \underbrace{\sum_k \sum_{c\neq k} \left(\sum_i h^{(k)}_{M-\delta+i} q^{(k)*}_i\right) \left(\sum_d h^{(c)*}_{M-\delta+d} q^{(c)}_d\right) \sum_{n,m=0}^{N-1} \sigma^4_s}_{A.2.1.} \\
& + \underbrace{\sum_{n,m=0}^{N-1} \sum_k \sum_{c\neq k} \left(\sum_i q^{(c)*}_i q^{(c)}_{m-n+i}\right) \left(\sum_j h^{(k)}_j h^{(k)*}_{m-n+j}\right) \sigma^4_s}_{A.3.1.} \\
& + \underbrace{\sum_{n,m=0}^{N-1} \sum_k \sum_{\ell\neq k} \left(\sum_i h^{(\ell)*}_{m-n+M-\delta+i} q^{(\ell)*}_i\right) \left(\sum_d h^{(k)}_{n-m+M-\delta+d} q^{(k)}_d\right) \sigma^4_s}_{A.4.1.\text{—only exists iff } s\in\mathbb{R}}
\end{aligned}
$$

Note that some terms only exist in case of real-valued sources. Expanding the terms in the previous equation,

$$
\begin{aligned}
&N^2\,\mathrm{E}\Big\{|\hat{\hat{h}}^{(0)}_{\nu+M-\delta}|^2\Big\} \\
&= \underbrace{N\sum_k \left|\sum_i h^{(k)}_{M-\delta+i} q^{(k)*}_i\right|^2 \kappa^{(k)}_s \sigma^4_s}_{A.1.1.} \;+\; \underbrace{N^2\sum_k \left|\sum_i h^{(k)}_{M-\delta+i} q^{(k)*}_i\right|^2 \sigma^4_s - N\sum_k \left|\sum_i h^{(k)}_{M-\delta+i} q^{(k)*}_i\right|^2 \sigma^4_s}_{A.1.2.} \\
&\quad \underbrace{+\sum_{n,m=0}^{N-1}\sum_k \left(\sum_i q^{(k)*}_i q^{(k)}_{m-n+i}\right)\left(\sum_j h^{(k)}_j h^{(k)*}_{m-n+j}\right)\sigma^4_s - \sum_{n,m=0}^{N-1}\sum_k \left(\sum_i q^{(k)*}_i q^{(k)}_{m-n+i} h^{(k)}_{M-\delta+i} h^{(k)*}_{m-n+M-\delta+i}\right)\sigma^4_s}_{A.1.3.} \\
&\quad \underbrace{+\sum_{n,m=0}^{N-1}\sum_k \left(\sum_i h^{(k)*}_{m-n+M-\delta+i} q^{(k)*}_i\right)\left(\sum_d h^{(k)}_{n-m+M-\delta+d} q^{(k)}_d\right)\sigma^4_s}_{A.1.4.\text{—only exists iff } s\in\mathbb{R}} \\
&\quad \underbrace{-\sum_{n,m=0}^{N-1}\sum_k \left(\sum_i h^{(k)*}_{m-n+M-\delta+i} q^{(k)*}_i h^{(k)}_{M-\delta+i} q^{(k)}_{m-n+i}\right)\sigma^4_s}_{A.1.4.\text{—only exists iff } s\in\mathbb{R}} \\
&\quad \underbrace{+N^2\left|\sum_k\sum_i h^{(k)}_{M-\delta+i} q^{(k)*}_i\right|^2 \sigma^4_s - N^2\sum_k \left|\sum_i h^{(k)}_{M-\delta+i} q^{(k)*}_i\right|^2 \sigma^4_s}_{A.2.1.} \\
&\quad \underbrace{+\sum_{n,m=0}^{N-1}\left(\sum_{c,i} q^{(c)*}_i q^{(c)}_{m-n+i}\right)\left(\sum_{k,j} h^{(k)}_j h^{(k)*}_{m-n+j}\right)\sigma^4_s - \sum_{n,m=0}^{N-1}\sum_k \left(\sum_i q^{(k)*}_i q^{(k)}_{m-n+i}\right)\left(\sum_j h^{(k)}_j h^{(k)*}_{m-n+j}\right)\sigma^4_s}_{A.3.1.} \\
&\quad \underbrace{+\sum_{n,m=0}^{N-1}\left(\sum_{\ell,i} h^{(\ell)*}_{m-n+M-\delta+i} q^{(\ell)*}_i\right)\left(\sum_{k,d} h^{(k)}_{n-m+M-\delta+d} q^{(k)}_d\right)\sigma^4_s}_{A.4.1.\text{—only exists iff } s\in\mathbb{R}} \\
&\quad \underbrace{-\sum_{n,m=0}^{N-1}\sum_k \left(\sum_i h^{(k)*}_{m-n+M-\delta+i} q^{(k)*}_i\right)\left(\sum_d h^{(k)}_{n-m+M-\delta+d} q^{(k)}_d\right)\sigma^4_s}_{A.4.1.\text{—only exists iff } s\in\mathbb{R}}
\end{aligned}
$$

A number of the terms cancel, leaving

$$
\begin{aligned}
&N^2\,\mathrm{E}\Big\{|\hat{\hat{h}}^{(0)}_{\nu+M-\delta}|^2\Big\} \\
&= N\sum_k \left|\sum_i h^{(k)}_{M-\delta+i} q^{(k)*}_i\right|^2 (\kappa^{(k)}_s-1)\sigma_s^4 \;-\sum_{n,m=0}^{N-1}\sum_k\left(\sum_i q^{(k)*}_i q^{(k)}_{m-n+i} h^{(k)}_{M-\delta+i} h^{(k)*}_{m-n+M-\delta+i}\right)\sigma_s^4 \\
&\quad \underbrace{-\sum_{n,m=0}^{N-1}\sum_k\left(\sum_i h^{(k)*}_{m-n+M-\delta+i} q^{(k)*}_i h^{(k)}_{M-\delta+i} q^{(k)}_{m-n+i}\right)\sigma_s^4}_{\text{iff } s\in\mathbb{R}} \\
&\quad + N^2\left|\sum_k\sum_i h^{(k)}_{M-\delta+i} q^{(k)*}_i\right|^2\sigma_s^4 + \sum_{n,m=0}^{N-1}\left(\sum_{c,i} q^{(c)*}_i q^{(c)}_{m-n+i}\right)\left(\sum_{k,j} h^{(k)}_j h^{(k)*}_{m-n+j}\right)\sigma_s^4 \\
&\quad \underbrace{+\sum_{n,m=0}^{N-1}\left(\sum_{\ell,i} h^{(\ell)*}_{m-n+M-\delta+i} q^{(\ell)*}_i\right)\left(\sum_{k,d} h^{(k)}_{n-m+M-\delta+d} q^{(k)}_d\right)\sigma_s^4}_{\text{iff } s\in\mathbb{R}}
\end{aligned}
$$

Recognizing the fourth term on the right side as $N^2|\hat{h}^{(0)}_{\nu+M-\delta}|^2$ and bringing it to the left,

$$
\begin{aligned}
\mathrm{E}\Big\{|\hat{\hat{h}}^{(0)}_{\nu+M-\delta}|^2\Big\} - \left|\hat{h}^{(0)}_{\nu+M-\delta}\right|^2 &= \frac{1}{N}\sum_k\left|\sum_i h^{(k)}_{M-\delta+i} q^{(k)*}_i\right|^2(\kappa^{(k)}_s-1)\sigma_s^4 \\
&\quad + \frac{1}{N^2}\sum_{n,m=0}^{N-1}\sum_{k,j} h^{(k)}_j h^{(k)*}_{m-n+j}\sum_{(c,i)\neq(k,j-M+\delta)} q^{(c)*}_i q^{(c)}_{m-n+i}\sigma_s^4 \\
&\quad \underbrace{+ \frac{1}{N^2}\sum_{n,m=0}^{N-1}\sum_{\ell,i} h^{(\ell)*}_{m-n+M-\delta+i} q^{(\ell)*}_i \sum_{(k,d)\neq(\ell,m-n+i)} h^{(k)}_{n-m+M-\delta+d} q^{(k)}_d \sigma_s^4}_{\text{iff } s\in\mathbb{R}}
\end{aligned}
$$

For general P, the previous equation has the form

$$
\begin{aligned}
\mathrm{E}\Big\{\big\|\hat{\hat{\mathbf{h}}}^{(0)}_{\nu+M-\delta}\big\|_2^2\Big\} - \big\|\hat{\mathbf{h}}^{(0)}_{\nu+M-\delta}\big\|^2 &= \frac{\sigma_s^4}{N}\sum_k\Big\|\sum_i \mathbf{h}^{(k)}_{M-\delta+i} q^{(k)*}_i\Big\|^2(\kappa^{(k)}_s-1) \\
&\quad + \frac{\sigma_s^4}{N^2}\sum_{n,m=0}^{N-1}\sum_{k,j}\mathbf{h}^{(k)H}_{m-n+j}\mathbf{h}^{(k)}_j\sum_{(c,i)\neq(k,j-M+\delta)} q^{(c)*}_i q^{(c)}_{m-n+i} \\
&\quad \underbrace{+ \frac{\sigma_s^4}{N^2}\sum_{n,m=0}^{N-1}\sum_{\ell,i}\mathbf{h}^{(\ell)H}_{m-n+M-\delta+i} q^{(\ell)*}_i\sum_{(k,d)\neq(\ell,m-n+i)}\mathbf{h}^{(k)}_{n-m+M-\delta+d} q^{(k)}_d}_{\text{iff } s\in\mathbb{R}}
\end{aligned}
$$

Combining the previous equation with (3) and (4) we find that

$$\begin{aligned}
\mathrm{E}\{\mathcal{E}_{\hat{\underline{\boldsymbol{h}}}}\} &= \mathcal{E}_{\hat{\underline{\boldsymbol{h}}}} + \frac{|\theta_{\hat{\underline{\boldsymbol{h}}}}|^2\sigma_s^4}{N(Q+1)}\sum_{\delta=0}^{Q}\Bigg(\sum_k \Big\|\sum_i \mathbf{h}_{M-\delta+i}^{(k)} q_i^{(k)*}\Big\|^2 (\kappa_s^{(k)}-1) \\
&\quad + \frac{1}{N}\sum_{n,m=0}^{N-1}\sum_{k,j}\mathbf{h}_{m-n+j}^{(k)H}\mathbf{h}_j^{(k)} \sum_{(c,i)\neq(k,j-M+\delta)} q_i^{(c)*} q_{m-n+i}^{(c)} \\
&\quad + \underbrace{\frac{1}{N^2}\sum_{n,m=0}^{N-1}\sum_{\ell,i}\mathbf{h}_{m-n+M-\delta+i}^{(\ell)H} q_i^{(\ell)*} \sum_{(k,d)\neq(\ell,m-n+i)} \mathbf{h}_{n-m+M-\delta+d}^{(k)} q_d^{(k)}\Bigg)}_{\text{iff } s\in\mathbb{R}}
\end{aligned} \tag{5}$$

Plugging (2) into the previous expression,

$$\begin{aligned}
\mathrm{E}\{\mathcal{E}_{\hat{\underline{\boldsymbol{h}}}}\} &= \mathcal{E}_{\hat{\underline{\boldsymbol{h}}}} + \frac{\sigma_s^4}{N(Q+1)}\frac{|\hat{\underline{\boldsymbol{h}}}_{\nu+M}^{(0)H}\underline{\boldsymbol{h}}_{\nu+M}^{(0)}|^2}{\big\|\hat{\underline{\boldsymbol{h}}}_{\nu+M}^{(0)}\big\|_2^4}\sum_{\delta=0}^{Q}\Bigg(\sum_k \Big\|\sum_i \mathbf{h}_{M-\delta+i}^{(k)} q_i^{(k)*}\Big\|^2 (\kappa_s^{(k)}-1) \\
&\quad + \frac{1}{N}\sum_{n,m=0}^{N-1}\Bigg(\sum_{k,j}\mathbf{h}_{m-n+j}^{(k)H}\mathbf{h}_j^{(k)}\Bigg)\Bigg(\sum_{(c,i)\neq(k,j-M+\delta)} q_i^{(c)*} q_{m-n+i}^{(c)}\Bigg) \\
&\quad + \underbrace{\frac{1}{N^2}\sum_{n,m=0}^{N-1}\Bigg(\sum_{\ell,i}\mathbf{h}_{m-n+M-\delta+i}^{(\ell)H} q_i^{(\ell)*}\Bigg)\Bigg(\sum_{(k,d)\neq(\ell,m-n+i)} \mathbf{h}_{n-m+M-\delta+d}^{(k)} q_d^{(k)}\Bigg)\Bigg)}_{\text{iff } s\in\mathbb{R}}
\end{aligned} \tag{6}$$