On Low-Complexity Estimation of Doubly-Selective Channels

Phil Schniter

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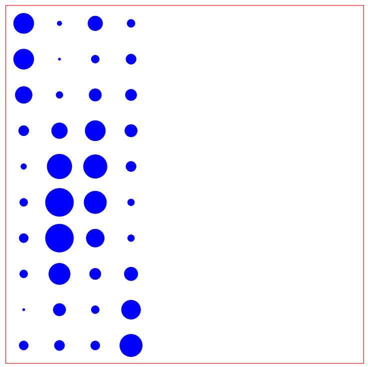
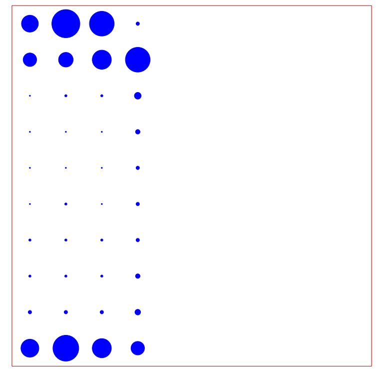
Doubly-Selective Channel Representations:

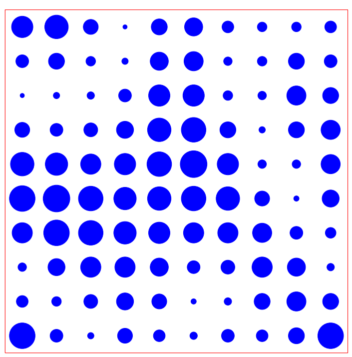
$\{h_{\text{tl}}(n,l)\}$: time/lag



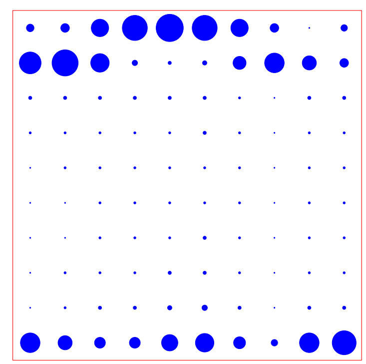
$\{h_{\text{dl}}(d,l)\}$: doppler/lag



$\{h_{\text{tf}}(,k)\}$: time/frequency

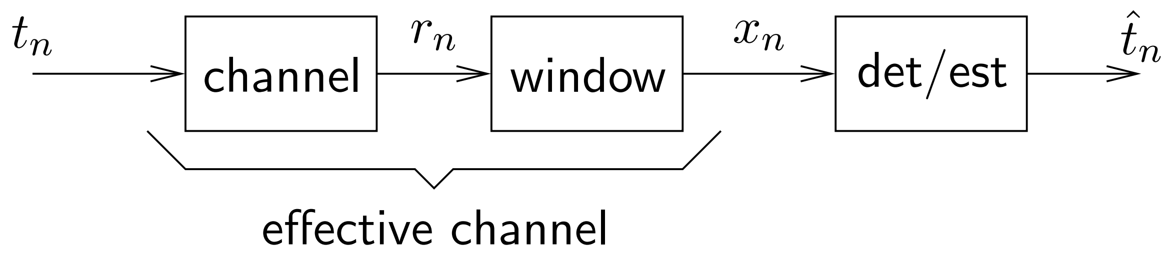


$\{h_{\text{df}}(d,k)\}$: doppler/frequency



$\rightarrow$ *doppler/lag is most compact, but still has residual leakage...*

Low Complexity Linear Pre-Processing:



For i^{th} data frame and window coefficients $\{b_n\}$,

$$x_n^{(i)} = b_n \left(\sum_{l=0}^{N_h-1} h_{\mathsf{tl}}^{(i)}(n,l) t_{n-l}^{(i)} + w_n^{(i)} \right)$$

$$= \sum_{l=0}^{N_h-1} \underbrace{b_n h_{\mathsf{tl}}^{(i)}(n,l)}_{\breve{h}_{\mathsf{tl}}^{(i)}(n,l)} t_{n-l}^{(i)} + \underbrace{b_n w_n^{(i)}}_{\text{colored noise}}$$

so <u>windowing the observation $\Leftrightarrow$ windowing the channel response.</u>

Note: Any finite-length processing of $\{r_n\}$ implies some sort of window!

A Doppler-Domain Interpretation:

- Time-domain signal:

$$\boldsymbol{r} \;=\; \mathcal{H}_{\text{tl}}\boldsymbol{t} + \boldsymbol{w}$$

- Windowed frequency-domain signal:

$$\begin{aligned}
\boldsymbol{x} \;=\; \boldsymbol{F}\,\mathcal{D}(\boldsymbol{b})\boldsymbol{r} &= \boldsymbol{F}\,\mathcal{D}(\boldsymbol{b})\mathcal{H}_{\text{tl}}\boldsymbol{t} + \boldsymbol{F}\,\mathcal{D}(\boldsymbol{b})\boldsymbol{w} \\
&= \underbrace{\boldsymbol{F}\,\mathcal{D}(\boldsymbol{b})\boldsymbol{F}^H}_{\mathcal{C}(\boldsymbol{\beta})}\underbrace{\boldsymbol{F}\mathcal{H}_{\text{tl}}}_{\mathcal{H}_{\text{dl}}}\boldsymbol{t} + \underbrace{\boldsymbol{F}\,\mathcal{D}(\boldsymbol{b})\boldsymbol{F}^H}_{\mathcal{C}(\boldsymbol{\beta})}\underbrace{\boldsymbol{F}\boldsymbol{w}}_{\boldsymbol{\nu}} \\
&\qquad \text{where } \boldsymbol{\beta} = \tfrac{1}{\sqrt{N}}\boldsymbol{F}\boldsymbol{b} \\
&= \underbrace{\mathcal{C}(\boldsymbol{\beta})\mathcal{H}_{\text{dl}}}_{\text{effective channel}}\boldsymbol{t} + \underbrace{\mathcal{C}(\boldsymbol{\beta})\boldsymbol{\nu}}_{\text{colored noise}}
\end{aligned}$$

so we are filtering in the Doppler domain.

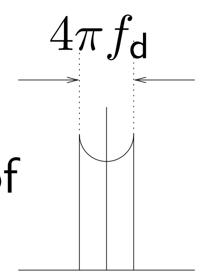
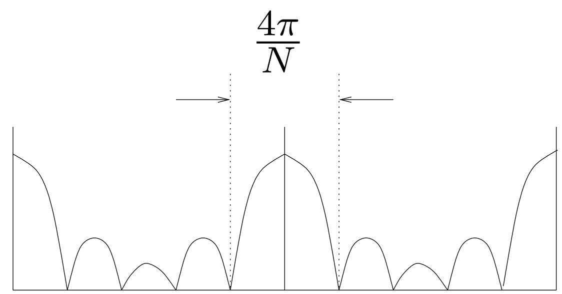
- Reminiscent of ISI shortening for MLSD state-reduction.

Effect of Window on Doppler Response:

Can show (assuming WSSUS):

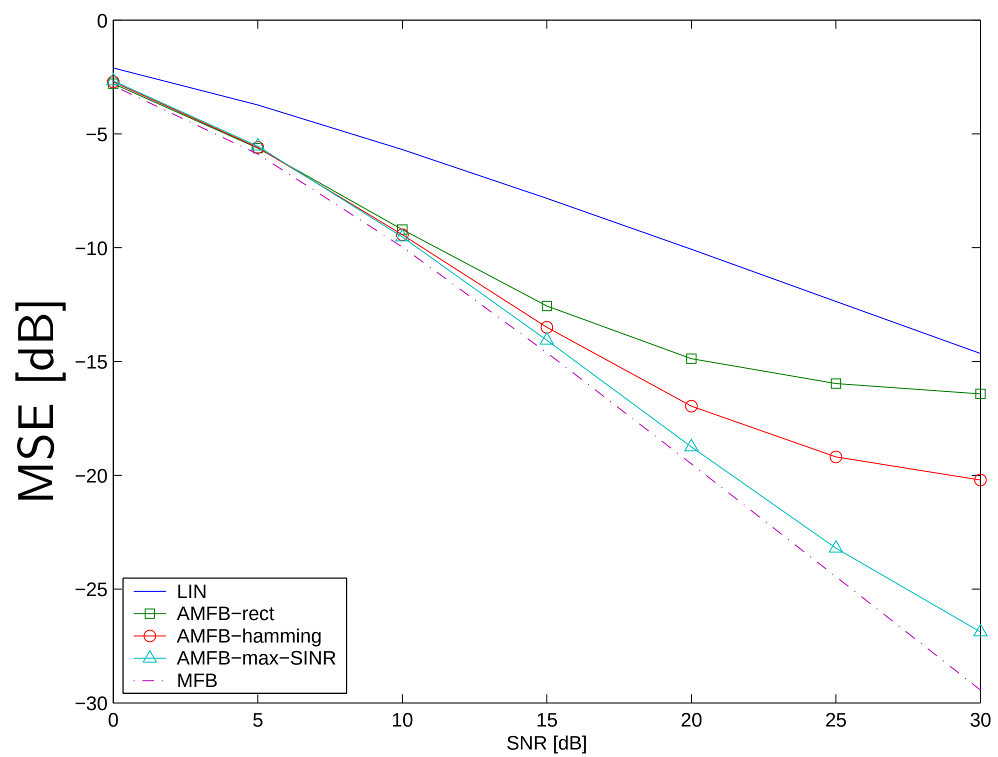
$$\mathrm{E}\Big\{|h_{\mathsf{dl}}(d,l)|^2\Big\} = \overbrace{S(\phi,l)}^{\text{Doppler spectrum}} * \overbrace{B(\phi)}^{\text{window spectrum}} \Bigg|_{\phi=\frac{2\pi}{N}k}$$

Illustrated below for Rayleigh fading and rectangular windowing:

$=$ Samples of  $*$ 

Note: *Zero Doppler spread* $\Rightarrow$ *Sample at sinc nulls* $\Rightarrow$ *Zero ICI*

Effect of Windowing on Symbol Estimation:

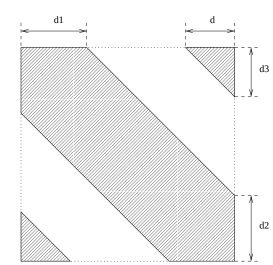


<u>CP-OFDM</u>

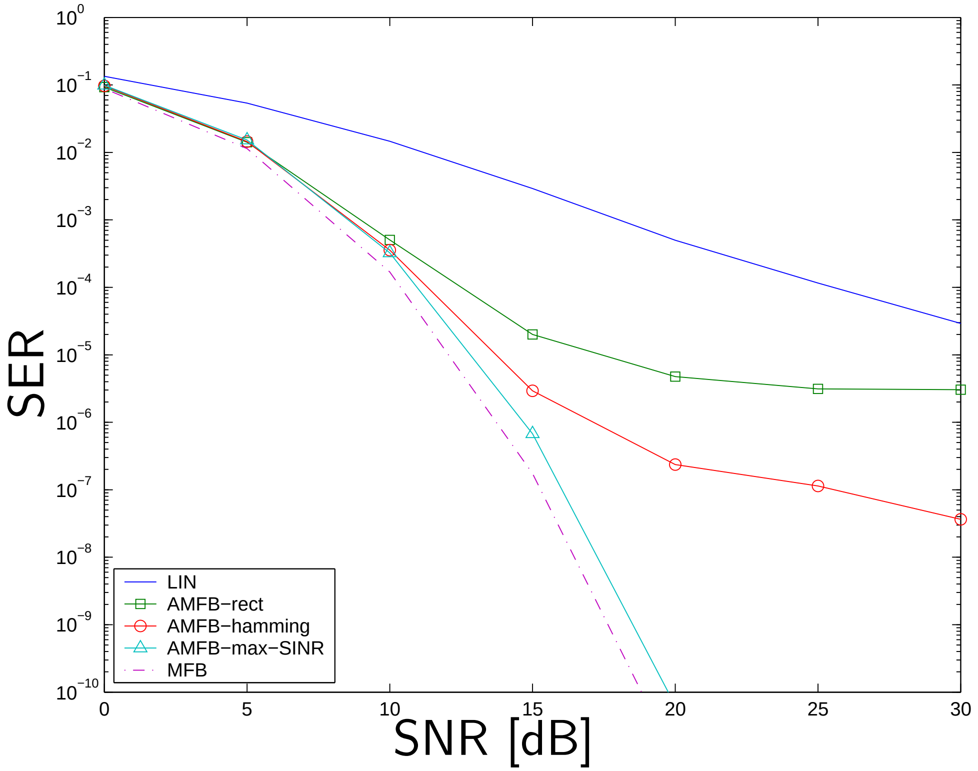
$$\boldsymbol{x} = \mathcal{H}_{\mathsf{df}} \boldsymbol{s} + \boldsymbol{\nu}$$

known $\mathcal{H}_{\mathsf{df}}$

$\mathcal{H}_{\mathsf{df}} \approx$



Effect of Windowing on Symbol Detection:



CP-OFDM

$$\boldsymbol{x} = \mathcal{H}_{\mathsf{df}}\boldsymbol{s} + \boldsymbol{\nu}$$

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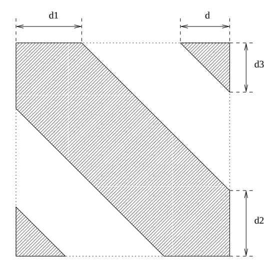
$\mathcal{H}_{\mathsf{df}} \approx$ 

Max-SINR Window Coefficients:

- Say $2D+1$ significant Doppler coefficients.
- Max-SINR window coefficients $\boldsymbol{b}_\star$ are

$$\boldsymbol{b}_\star \;=\; \text{gen-evec}_{\text{max}}\Big(\boldsymbol{A}\odot\boldsymbol{R}^*,\;\; \text{diag}(\boldsymbol{R}+\sigma^2\boldsymbol{I})-\boldsymbol{A}\odot\boldsymbol{R}^*\Big)$$

where, for WSSUS Rayleigh fading,

$$[\boldsymbol{A}]_{m,n} \;=\; \frac{\sin\big(\frac{\pi}{N}(2D+1)(n-m)\big)}{N\sin\big(\frac{\pi}{N}(n-m)\big)}$$

$$[\hat{\boldsymbol{R}}]_{n,m} \;=\; J_0\big(2\pi f_{\mathsf{d}}(n-m)\big)\sum_{l=0}^{N_h-1}\sigma_l^2$$

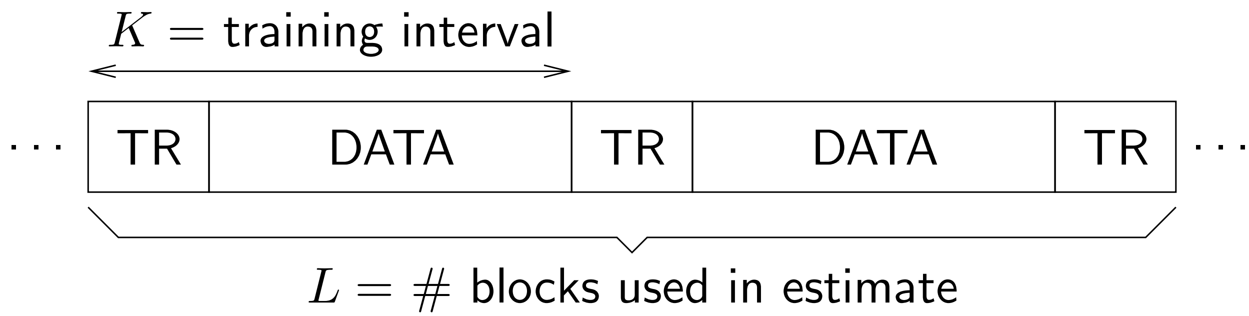
- Note that $\boldsymbol{b}_\star$ is a function of $\Big\{D, N, f_{\mathsf{d}}, \frac{\sum\sigma_l^2}{\sigma^2}\Big\}$

Relationship to Channel Estimation:

- Parsimonious representation $\Rightarrow$ few channel parameters to estimate.
- E.g., $\boldsymbol{H}_{\mathsf{dl}}$ has $2DN_h$ significant parameters, $D \approx f_{\mathsf{d}}N$.
- Given $\boldsymbol{H}_{\mathsf{dl}}$ could compute, if necessary,
  - $\boldsymbol{H}_{\mathsf{tl}}$ via N_h FFTs of length N,
  - $\boldsymbol{H}_{\mathsf{df}}$ via $2D$ FFTs of length N.

$\rightsquigarrow$ *Can we estimate $\boldsymbol{H}_{\mathsf{dl}}$ in a computationally efficient manner?*

Training-based Approaches:



We'll focus on two ideas:

- Complex-exponential bursts of length N_h.

  $\rightsquigarrow$ FFT-based channel estimation

- Kronecker delta bursts of length $2N_h - 1$.

  $\rightsquigarrow$ interpolation-based channel estimation

The former is novel while the latter is well-known.

FFT-Based Channel Estimation:

$$h_{\mathsf{tl}}(n,d) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} h_{\mathsf{dl}}^{(i)}(k,d) e^{j\frac{2\pi}{N}kn} \quad \text{for } n = i,\ldots,i+N-1$$

$$x_n = w_n + \sum_{d=0}^{N_h-1} h_{\mathsf{tl}}(n,d) t_{n-d} \quad \text{(windowing omitted)}$$

$$= w_n + \boldsymbol{f}_n^H \boldsymbol{H}_{\mathsf{dl}}^{(i)} \boldsymbol{t}_n$$

where

$$\begin{cases} \boldsymbol{f}_n^H := \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi}{N}n\cdot 0} & \cdots & e^{j\frac{2\pi}{N}n(N-1)} \end{bmatrix} \\ \boldsymbol{H}_{\mathsf{dl}}^{(i)} := \begin{bmatrix} h_{\mathsf{dl}}^{(i)}(0,0) & \ldots & h_{\mathsf{dl}}^{(i)}(0,N_h-1) \\ \vdots & & \vdots \\ h_{\mathsf{dl}}^{(i)}(N-1,0) & \ldots & h_{\mathsf{dl}}^{(i)}(N-1,N_h-1) \end{bmatrix} \\ \boldsymbol{t}_n := \begin{bmatrix} t_n & \cdots & t_{n-N_h+1} \end{bmatrix}^t. \end{cases}$$

FFT-Based Channel Estimation (cont.):

Next, separate the $2D$ "big" and $(N-2D)$ "small" rows of $\boldsymbol{H}_{\mathsf{dl}}^{(i)}$:

$$\begin{aligned} x_n &= w_n + \boldsymbol{f}_n^H \boldsymbol{H}_{\mathsf{dl}}^{(i)} \boldsymbol{t}_n \\ &= w_n + \boldsymbol{f}_{\mathsf{b},n}^H \boldsymbol{H}_{\mathsf{dl,b}}^{(i)} \boldsymbol{t}_n + \boldsymbol{f}_{\mathsf{s},n}^H \boldsymbol{H}_{\mathsf{dl,s}}^{(i)} \boldsymbol{t}_n \\ &= w_n + (\boldsymbol{t}_n^t \otimes \boldsymbol{f}_{\mathsf{b},n}^H) \boldsymbol{h}_{\mathsf{dl,b}}^{(i)} + (\boldsymbol{t}_n^t \otimes \boldsymbol{f}_{\mathsf{s},n}^H) \boldsymbol{h}_{\mathsf{dl,s}}^{(i)} \end{aligned}$$

where $\boldsymbol{h}_{\mathsf{dl,b}}^{(i)} = \mathsf{vec}(\boldsymbol{H}_{\mathsf{dl,b}}^{(i)})$ & $\boldsymbol{h}_{\mathsf{dl,s}}^{(i)} = \mathsf{vec}(\boldsymbol{H}_{\mathsf{dl,s}}^{(i)})$, and stack L observations:

$$\begin{aligned} \underline{\mathbf{x}}^{(i)} &:= \begin{bmatrix} \vdots \\ x_n \\ \vdots \end{bmatrix}, \quad \boldsymbol{T}_{\mathsf{b}}^{(i)} := \begin{bmatrix} \vdots \\ \boldsymbol{t}_n^t \otimes \boldsymbol{f}_{\mathsf{b},n}^H \\ \vdots \end{bmatrix}, \quad \boldsymbol{T}_{\mathsf{s}}^{(i)} := \begin{bmatrix} \vdots \\ \boldsymbol{t}_n^t \otimes \boldsymbol{f}_{\mathsf{s},n}^H \\ \vdots \end{bmatrix}, \\ \underline{\mathbf{x}}^{(i)} &= \boldsymbol{T}_{\mathsf{b}}^{(i)} \boldsymbol{h}_{\mathsf{dl,b}}^{(i)} + \underbrace{\boldsymbol{T}_{\mathsf{s}}^{(i)} \boldsymbol{h}_{\mathsf{dl,s}}^{(i)} + \underline{\mathbf{w}}^{(i)}}_{\text{interference + noise}}. \end{aligned}$$

Consider Zero-forcing and MMSE approaches to estimation of $\boldsymbol{h}_{\mathsf{dl,b}}^{(i)}$...

Zero-Forcing FFT-Based Channel Estimation:

Idea:

$$\widehat{\boldsymbol{h}^{(i)}_{\mathsf{dl,b}}} \;=\; \left(\boldsymbol{T}^{(i)}_{\mathsf{b}}\right)^{-1}\underline{\mathbf{x}}^{(i)}$$

To make $\boldsymbol{T}_{\mathsf{b}}$ easy to invert, choose

$$\begin{aligned}
L &= 2DN_h \quad \text{(\# observations)}\\
K &= N/L \quad \text{(sampling interval)}\\
\underline{\mathbf{x}}^{(i)} &= \begin{bmatrix} x_{iK} & x_{(i+1)K} & x_{(i+2)K} & \cdots & x_{(i+L-1)K}\end{bmatrix}^t\\
\boldsymbol{t}_{iK} &= e^{j\frac{2\pi}{N}DiK}\begin{bmatrix} e^{j\frac{2\pi}{N}2DiK\cdot 0} & \cdots & e^{j\frac{2\pi}{N}2DiK\cdot(N_h-1)}\end{bmatrix}^t,
\end{aligned}$$

so that

$$\boldsymbol{T}^{(iK)}_{\mathsf{b}} = \frac{1}{\sqrt{K}}\boldsymbol{F}^H_L \operatorname{diag}\left(\begin{bmatrix} e^{j\frac{2\pi}{L}i\cdot 0} & \cdots & e^{j\frac{2\pi}{L}i(L-1)}\end{bmatrix}\right).$$

$$\Rightarrow \quad \boxed{\widehat{\boldsymbol{h}^{(i)}_{\mathsf{dl,b}}} \;=\; \sqrt{K}\operatorname{diag}\left(\begin{bmatrix} e^{-j\frac{2\pi}{L}i\cdot 0} & \cdots & e^{-j\frac{2\pi}{L}i(L-1)}\end{bmatrix}\right)\boldsymbol{F}_L\underline{\mathbf{x}}^{(iK)}}$$

MMSE FFT-Based Channel Estimation:

For better performance, try

$$\widehat{\boldsymbol{h}^{(i)}_{\mathsf{dl,b}}} = \mathrm{E}\{\boldsymbol{h}^{(iK)}_{\mathsf{dl,b}} \underline{\mathbf{x}}^{(iK)H}\} \, \mathrm{E}\{\underline{\mathbf{x}}^{(iK)} \underline{\mathbf{x}}^{(iK)H}\}^{-1} \underline{\mathbf{x}}^{(iK)}$$

It can be shown that (WSSUS, Rayleigh, uniform power profile):

$$\widehat{\boldsymbol{h}^{(i)}_{\mathsf{dl,b}}} = \underbrace{\boldsymbol{D}^{(iK)H}}_{L \text{ mults}} \cdot \underbrace{\left(\boldsymbol{A}^H \odot \boldsymbol{F}_L\right)}_{DL \log L \text{ mults}} \cdot \underbrace{\mathsf{vec}\Big(\mathsf{mat}(\underline{\mathbf{x}}^{(iK)}) \boldsymbol{R}_x^{-1}\Big)}_{2DL \text{ mults}}$$

where

$$\boldsymbol{D}^{(iK)} = \tfrac{1}{\sqrt{K}} \operatorname{diag}\left(\left[e^{-j\frac{2\pi}{L} i \cdot 0} \quad \cdots \quad e^{-j\frac{2\pi}{L} i(L-1)} \right]\right)$$

$$[\boldsymbol{A}]_{p,q} = \frac{1}{N_h} \sum_{m=pK-N+1}^{pK} J_0(2\pi f_{\mathsf{d}} m) e^{-j\frac{2\pi}{N}(\langle q \rangle_{2D} - D)m}, \quad \underline{2D \text{ unique cols}}$$

$$[\boldsymbol{R}_x]_{i+q,i} = \sigma_w^2 \delta(i) + (-1)^q J_0(\pi f_{\mathsf{d}} N D^{-1} q), \quad \boldsymbol{R}_x \in \mathbb{C}^{2D \times 2D}$$

Limitations:

With the FFT-based schemes:

$$\begin{aligned} D &\approx f_{\mathsf{d}} N \\ K &= \frac{N}{2DN_h} \\ K &\geq N_h \end{aligned}$$

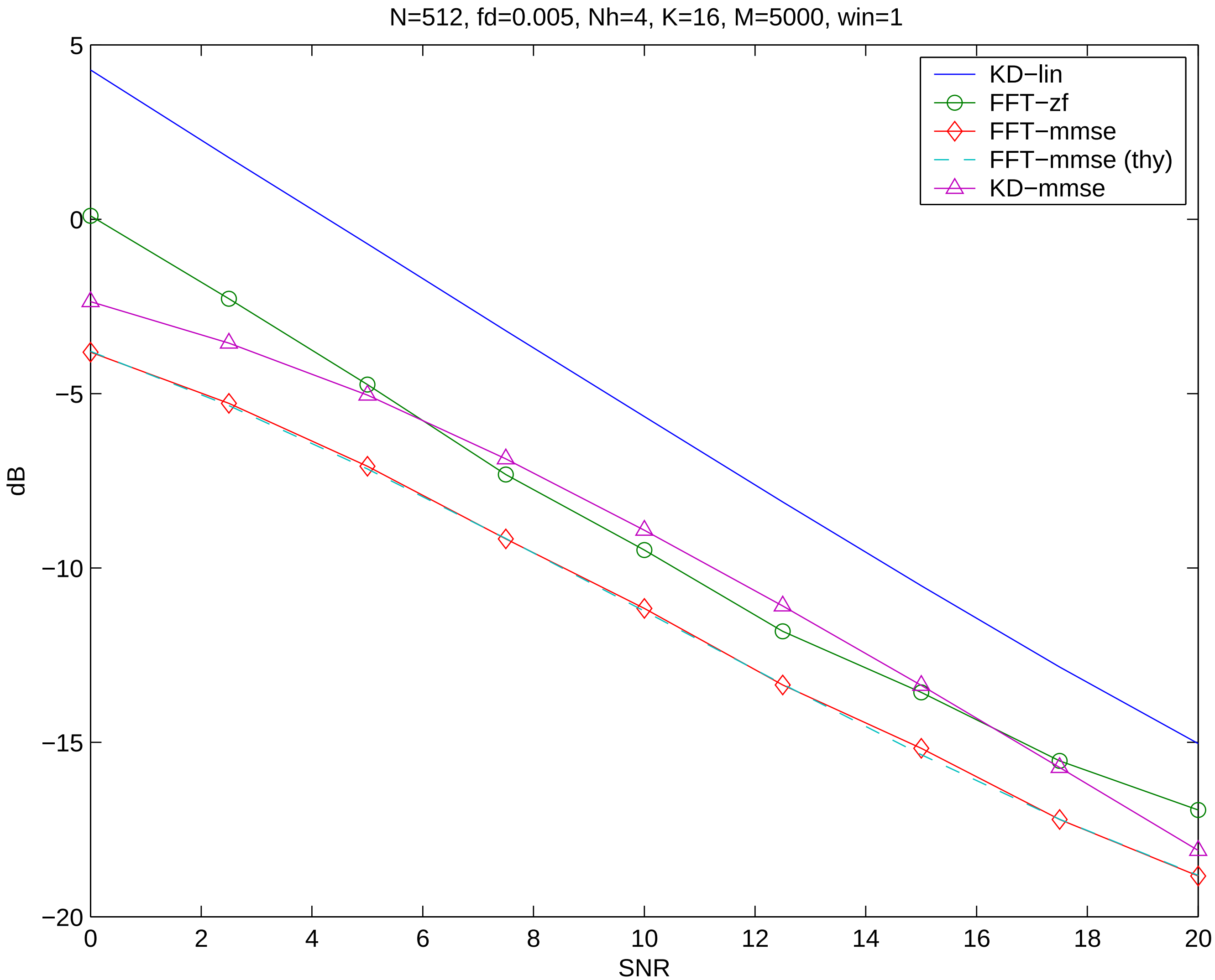
we find

$$f_{\mathsf{d}}\Big|_{\max} \approx \frac{D}{N} = \frac{1}{2KN_h} \leq \frac{1}{2N_h^2}$$

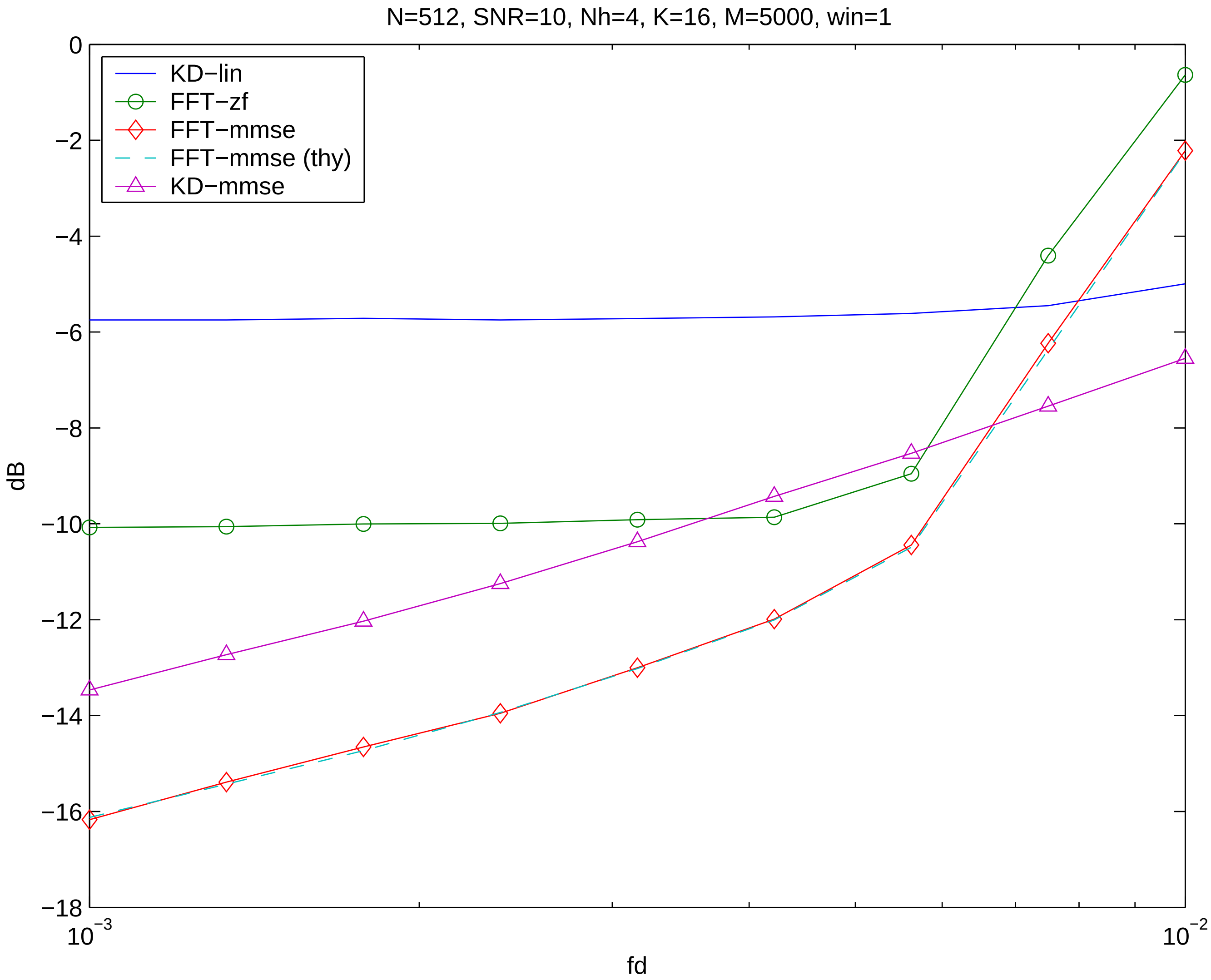
giving a limitation on $\{f_{\mathsf{d}}, N_h\}$.

(Perhaps a different construction could get around this?)

Numerical Results (MSE vs. SNR):



Numerical Results (MSE vs. Doppler):



Conclusions:

- Parsimonious $\{h_{\mathsf{dl}}(n,l)\}$ channel representation:

  $$L = 2DN_h \text{ significant coefficients.}$$

- Windowing can further squeeze channel into this representation.

- With careful pilot selection, $\{h_{\mathsf{dl}}(n,l)\}$ can be estimated via
  - FFT-ZF scheme: one FFT of length L
  - FFT-MMSE scheme: $2D$ FFTs of length L

- Compared to KD-MMSE-interpolation method...
  - lower complexity: $\frac{L}{2}\log L$ or $DL\log L$ versus $2L^2N_h$ ops.
  - comparable or better performance.
  - limited to low $f_{\mathsf{d}}N_h$.

    $\rightsquigarrow$ *currently looking for a way around this...*